A geometric construction of tight Gabor frames with multivariate compactly supported smooth windows

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November 1, 2010

Abstract

The geometry of fundamental domains of lattices was used by Han and Wang to construct multivariate Gabor frames for separable lattices. We build upon their results to obtain Gabor frames with smooth and compactly supported window functions. For this purpose we study pairs of lattices which have equal density and allow for a common compact and star-shaped fundamental domain. The results are then extended to a larger class of lattices via symplectic equivalence. [1]

1 Introduction

Gabor systems are useful to construct discrete time-frequency representations of signals. A Gabor system is customarily denoted as $(g, \Lambda) = \{M_\omega T_x g : (x, \omega) \in \Lambda\}$, where the square integrable function g is the Gabor window, $\Lambda = M\mathbb{Z}^{2d}$, M full rank, a lattice in the time-frequency space $\mathbb{R}^{2d}$, T_x the translation operator $(T_x f)(t) = f(t-x)$, $x \in \mathbb{R}^d$, and M_ω the modulation operator $(M_\omega f)(t) = e^{2\pi i\langle \omega, y\rangle} f(t)$, $\omega \in \mathbb{R}^d$. A Gabor system is a tight frame for the space of square integrable functions $L^2(\mathbb{R}^d)$ if up to a scalar factor, Parseval's identity holds. That is, for some $c > 0$, we have

$$f = c \sum_{(x,\omega)\in\Lambda} \langle f, M_\omega T_x g\rangle M_\omega T_x g, \quad f \in L^2(\mathbb{R}^d), \tag{1}$$

where the series converges unconditionally and the computation of the representation coefficients is stable.

[1] *Math Subject Classifications:* 42C15, 42C40

Keywords and phrases: time-frequency lattices; lattice tilings and packings; symplectic equivalence.

Necessary and sufficient conditions for the Gabor system (g, Λ) to be a frame for $L^2(\mathbb{R}^d)$ have a long history in applied harmonic analysis. A classical statement is the density condition, which states that any Gabor system whose lattice $\Lambda = M\mathbb{Z}^{2d}$ has density $d(\Lambda) = 1/|\det M| < 1$ is not complete and therefore not a Gabor frame for $L^2(\mathbb{R}^d)$ [Dau90, Dau92, Jan94]. Recently Bekka showed that the density condition is sufficient as well: for each Λ with $d(\Lambda) \geq 1$ there exists a function $g \in L^2(\mathbb{R}^d)$ such that (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ [Bek04]. In his fundamental paper, Bekka proves existence only and his work reveals nothing about the window besides membership in $L^2(\mathbb{R}^d)$. Multivariate Gabor frames are constructed more explicitly for separable lattices in [HW01, HW04], however, the obtained Gabor windows are characteristic functions on sets that are fundamental domains for pairs of lattices. These fundamental domains may well be unbounded; in this case the constructed Gabor windows decay neither in time nor in frequency, so (1) is not local. For any lattice Λ, the existence of so-called multi-window Gabor frames with windows in the Schwarz space or in modulation spaces has been shown in [Rie88, Lue09], but the number of windows needed does not follow from their analysis, for example, it is not clear whether $d(\Lambda) > 1$ implies that a single window in the Schwarz space suffices.

In this paper, we further explore the results of Han and Wang and use our findings to construct tight Gabor frames (g, Λ) in $L^2(\mathbb{R}^2)$ for separable lattices $\Lambda = \Lambda_1 \times \Lambda_2 \subset \mathbb{R}^4$ with nonnegative, smooth and compactly supported windows, that is, $g \in C_c^\infty(\mathbb{R}^2)$. To achieve this, we shall assume star-shapedness of the common fundamental domain of a pair of lattices. Note that the so-called Balian-Low theorem [BHW98, GHHK02] implies that if (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ with g being in the Schwarz space $\mathcal{S}(\mathbb{R}^d)$, then $d(\Lambda) > 1$. Hence, our analysis is limited to lattices with density greater than 1.

The characterization of lattices Λ so that (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ for a fixed g are rare in the literature [Lyu92, SW92, JS02, Jan03], and with the exception of [Grö10, PR10] these concern the case $d = 1$ only. In principle, characterizing spanning properties of Gabor frames for lattices Λ of higher dimension ($\Lambda \subset \mathbb{R}^{2d}, d \geq 2$) is much more intricate than in the one-dimensional case.

The paper is organized as follows. Section 2 recalls basic facts from Gabor analysis and from the Fourier theory of translational tilings that are used in this paper. Section 3 contains results from [HW01, HW04] as well as minor extensions of these. The construction of smooth and compactly supported Gabor windows for a class of separable lattices is presented in Section 4. These results extend to some lower block-triangular matrices, Section 5 contains examples in the bivariate case, that is, we present pairs of lattices in $\mathbb{R}^4$ which allow for a common star-shaped fundamental domain.

2 Tools in time-frequency analysis

Throughout the text, all subsets of $\mathbb{R}^d$ are assumed to be Lebesgue measurable. For $\Omega \subseteq \mathbb{R}^d$, we use $\Omega + x = \{y + x : \ y \in \Omega\}$, $x \in \mathbb{R}^d$, $\gamma\Omega = \{\gamma y : \ y \in \Omega\}$, $\gamma > 0$, and $\Omega_\epsilon = \{y + z : \ y \in \Omega, \ \|z\|_2 < \epsilon\}$, $\epsilon > 0$, where $\| \ \|_2$ denotes the Euclidean norm on $\mathbb{R}^d$. The characteristic function on $\Omega \subseteq \mathbb{R}^d$ is denoted by χ_Ω, that is, we have $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ else.

We use the standard notation for complex-valued functions on $\mathbb{R}^d$: $\mathcal{S}(\mathbb{R}^d)$ is the Schwarz space, and $C_c^\infty(\mathbb{R}^d)$ the space of smooth and compactly supported functions.

A *time-frequency shift* is $(\pi(\lambda)f)(t) = (M_\omega T_x f)(t) = f(t-x)e^{2\pi i \langle \omega, t\rangle}$ for $\lambda = (x, \omega) \in \mathbb{R}^{2d}$. The Fourier transformation used here is $L^2(\mathbb{R}^d)$–normalized, that is, $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(y) e^{-2\pi i \xi \cdot y} dy$ for f integrable.

Definition 2.1 *A lattice Λ in $\mathbb{R}^{2d}$ is a discrete subgroup of the additive group $\mathbb{R}^{2d}$, that is $\Lambda = M\mathbb{Z}^d$, with M being full-rank ($\det M \neq 0$). A separable lattice has the form $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$. The dual lattice of $\Lambda = M\mathbb{Z}^d$ is $\Lambda^\perp = M^{-T}\mathbb{Z}^{2d}$, and the adjoint lattice of $\Lambda = M\mathbb{Z}^{2d}$ is $\Lambda^\circ = \{\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d : \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda)$ for all $\mu \in \Lambda\}$. The volume of the lattice $\Lambda = M\mathbb{Z}^d$ equals the Lebesgue measure of $\mathbb{R}^d/\Lambda$, that is $\operatorname{vol}\Lambda = m(\mathbb{R}^d/\Lambda) = |\det M|$, and the density of Λ is $d(\Lambda) = (\operatorname{vol}\Lambda)^{-1}$.*

We have $d(\Lambda^\perp) = d(\Lambda^\circ) = 1/d(\Lambda)$, and the adjoint Λ° of a separable lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ is $\Lambda^\circ = (B\mathbb{Z}^d)^\perp \times (A\mathbb{Z}^d)^\perp = (B^{-T}\mathbb{Z}^d) \times (A^{-T}\mathbb{Z}^d)$, where M^{-T} denotes $(M^T)^{-1}$. Moreover, $(M^\circ)^\circ = M$ and $(M^\perp)^\perp = M$ [Grö01].

Definition 2.2 *Let $\Omega \subset \mathbb{R}^d$ be measurable and let Λ be a full rank lattice. If $(\Omega + \lambda_1) \cap (\Omega + \lambda_2)$ is a null set for $\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \Lambda$, then Ω is a packing set for Λ. If in addition, $\mathbb{R}^d = \cup_{\lambda \in \Lambda}(\Omega + \lambda)$, then Ω is a tiling set (fundamental domain) for Λ.*

Clearly, a measurable set Ω is a packing set if and only if $\sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) \leq 1$ for almost all $x \in \mathbb{R}^d$ and a fundamental domain for Λ if and only if $\sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) = 1$ for almost all $x \in \mathbb{R}^d$. Furthermore, if Ω is a packing set and $m(\Omega) = \operatorname{vol}\Lambda$, then Ω is a fundamental domain for Λ. Moreover, any translate of a fundamental domain of Λ is a fundamental domain of Λ.

If Ω is a star-shaped fundamental domain for Λ, that is, there exists a point $N \in \Omega$ such that for all $Q \in \Omega$ the line segment $\overrightarrow{NQ}$ is contained entirely within Ω (see Figure 1), then $\gamma(\Omega - Q)$ is a packing set for Λ for all $0 < \gamma \leq 1$.

Fundamental domains for lattices can be characterized with methods from Fourier analysis [Kol04].

Lemma 2.3 *Let Λ be a lattice in $\mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ be measurable. Ω is a fundamental domain for Λ if and only if $\widehat{\chi_\Omega}$ vanishes on $\Lambda^\perp \backslash \{0\}$.*

Moreover, the following result is central to our analysis [Fug74, IKT03, KM06].

Theorem 2.4 *The set Ω is a fundamental domain of the lattice Λ in $\mathbb{R}^d$ if and only if the set of pure frequencies $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda^\perp}$ is an orthonormal basis for $L^2(\Omega)$.*

We recall some important definitions and properties from the theory of Gabor frames and Gabor Riesz basis sequences.

Definition 2.5 *A Gabor system (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ with frame bounds $0 < a \leq b$ if*

$$a\|f\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq b\|f\|_{L^2}^2 \quad \text{for all } f \in L^2(\mathbb{R}^d). \tag{2}$$

The Gabor frame is tight if $a = b$ is possible. A Gabor system (g, Λ) is a Riesz basis for $L^2(\mathbb{R}^d)$ if there exist constants $0 < a \leq b$ such that

$$a\|\mathbf{c}\|_{\ell^2}^2 \leq \Big\| \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g \Big\|_{L^2}^2 \leq b\|\mathbf{c}\|_{\ell^2}^2 \quad \text{for all } \mathbf{c} \in \ell^2(\Lambda), \tag{3}$$

and $\overline{\operatorname{span}(g, \Lambda)} = L^2(\mathbb{R}^d)$. A Gabor Riesz basis sequence is a Riesz basis for the L^2-closure of its linear span.

Clearly, if (g, Λ) is a Riesz basis sequence with $\|g\|_{L^2} = 1$ and $a = b = 1$ in (3) then (g, Λ) is is an orthonormal sequence. Moreover, we shall use below that if (g, Λ) is a tight Gabor frame, then $a = b = d(\Lambda)\|g\|_{L^2}^2$ in (2) and $c = (d(\Lambda)\|g\|_{L^2}^2)^{-1}$ in (1) (Theorem 5 in [BCHL06]).

The usefulness of Gabor frames and tight Gabor frames stems from the following result.

Theorem 2.6 *Let $g \in L^2(\mathbb{R}^d)$ and let Λ be a full rank lattice. If (g, Λ) is a frame for $L^2(\mathbb{R}^d)$, then exists a so-called dual window $\gamma \in L^2(\mathbb{R}^d)$ with*

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda) g = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)\gamma, \quad f \in L^2(\mathbb{R}^d)\,. \tag{4}$$

If (g, Λ) is a tight frame for $L^2(\mathbb{R}^d)$, then we can choose in (4) $\gamma = (d(\Lambda)\|g\|_{L^2}^2)^{-1}\, g$, that is, (1) holds with $c = (d(\Lambda)\|g\|_{L^2}^2)^{-1}$.

The benefit of having compactly supported and smooth g in (1) and (4) is clear. Only then, we can guarantee that for any $r \in \mathbb{N}$ and any compactly supported r-times differentiable function f, there exists $N_f, C_f > 0$ with $\langle f, \pi(x,\omega) g\rangle = 0$ whenever $\|x\|_2 \geq N_f$, and $|\langle f, \pi(x,\omega) g\rangle| \leq C|\omega|^{-r}$. This property also allows for efficient quantization of the expansion coefficients in (1) and (4) [Yıl03].

For a detailed discussion of Bessel sequences, Gabor frames and Riesz basis sequences, we refer to [Grö01, Chr03].

The following results are central in the theory of Gabor frames [RS97, FG97, FZ98].

Theorem 2.7 *Let $g \in L^2(\mathbb{R}^d)$ and let Λ be a full rank lattice. Then (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ if and only if (g, Λ°) is a Riesz sequence. Moreover, (g, Λ) is a tight frame for $L^2(\mathbb{R}^d)$ if and only if (g, Λ°) is an orthonormal sequence.*

Gabor frame theory is rooted in the representation theory of the Weyl-Heisenberg group, a fact that we shall exploit in Sections 3 and 4 [Fol89, Grö01]. In particular, we shall use so-called metaplectic operators which are discussed below.

Definition 2.8 *The symplectic group* $\mathrm{Sp}(d)$ *is the subgroup of $GL(2d, \mathbb{R})$ whose members $\left(\begin{smallmatrix} A & C \\ D & B \end{smallmatrix}\right)$ are characterized by $AD^T = A^T D$, $BC^T = B^T C$ and $A^T B - D^T C = I$. A lattice $\Lambda = M\mathbb{Z}^{2d}$ with $M \in \mathrm{Sp}(d)$ is called symplectic lattice.*

Theorem 2.9 *For $M \in \mathrm{Sp}(d)$ exists a unitary operator $\mu(M)$ on $L^2(\mathbb{R}^d)$, a so-called metaplectic operator, with $\pi(M\lambda) = \mu(M)^*\pi(\lambda)\mu(M)$, $\lambda \in \mathbb{R}^{2d}$.*

Theorem 2.10 *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ and M be a symplectic matrix in $GL(\mathbb{R}, 2d)$. Then the following are equivalent:*

1. *There exists $g \in L^2(\mathbb{R}^d)$, respectively $g \in \mathcal{S}(\mathbb{R}^d)$, such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$.*

2. *There exists $\tilde{g} \in L^2(\mathbb{R}^d)$, respectively $\tilde{g} \in \mathcal{S}(\mathbb{R}^d)$, such that $(\tilde{g}, M\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$.*

Theorem 2.10 follows from Theorem 2.9 and the choice $\tilde{g} = \mu(M)g$. In fact, all metaplectic operators restrict to $\mathcal{S}(\mathbb{R}^d)$ [Grö01], but unfortunately not to $C_c^\infty(\mathbb{R}^d)$ as discussed in the next paragraph. In general, the spanning properties of the Gabor system (g, Λ) are transferred to the Gabor system $(\mu(M)g, M\Lambda)$ [Grö01]. Thus, we can replace the quantifier 'a Gabor frame for $L^2(\mathbb{R}^d)$' by 'Riesz basis sequence' or 'frame sequence'.

Note that $\mathrm{Sp}(d)$ is generated by matrices of the form $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} B & 0 \\ 0 & B^{-T} \end{smallmatrix}\right)$, $\det B \neq 0$, and $\left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$, C - symmetric. The corresponding metaplectic operators are, respecitively, the Fourier transform, the normalized dilation $f \mapsto |\det B|^{1/d} f \circ B$, and the multiplication with a chirp $e^{\pi i\langle x, Cx\rangle}$. Clearly, $\mu(M)$ restricts to $C_c^\infty(\mathbb{R}^d)$ if M is generated by matrices of the form $\left(\begin{smallmatrix} B & 0 \\ 0 & B^{-T} \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$, C - symmetric, a fact that we shall exploit below.

3 Characteristic functions as Gabor frame windows

Han and Wang construct Gabor systems with windows that are characteristic functions on fundamental domains of pairs of lattices [HW01, HW04]. Their construction is based on part 1 implies part 3 of the following result.

Proposition 3.1 *Let $\Omega \subseteq \mathbb{R}^d$ be a fundamental domain for $A\mathbb{Z}^d$. The following are equivalent.*

1. *Ω is a packing set for $B^{-T}\mathbb{Z}^d$;*
2. *$(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a frame for $L^2(\mathbb{R}^d)$;*
3. *$(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound $1/|\det B|$;*
4. *$(\chi_\Omega, B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d)$ is a Riesz basis sequence;*
5. *$((m(\Omega)^{-1/2}\chi_\Omega, B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d)$ is an orthonormal sequence.*

Note that Ω being a fundamental domain for $A\mathbb{Z}^d$ and a packing set for $B^{-T}\mathbb{Z}^d$ implies $|\det B^{-T}| \geq |\det A|$ and for $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$, we then have $d(\Lambda) \geq 1$ and for $\Lambda = B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d$, we have $d(\Lambda) \leq 1$.

Proof. We shall show that part 1 implies part 5, and part 4 implies part 1. Clearly, part 3 implies part 2 and part 5 implies part 4. The equivalence of parts 2 and 4 and parts 3 and 5 follows from Theorem 2.7. The explicit frame bound in part 3 follows from $\|\chi_\Omega\|_{L^2}^2 = |\det A|$ and $d(A\mathbb{Z}^d \times B\mathbb{Z}^d) = 1/|\det A \det B|$ [BCHL06].

Note that Ω being a packing set for $B^{-T}\mathbb{Z}^d$ implies

$$\langle \pi(x,\omega)\chi_\Omega, \pi(x',\omega')\chi_\Omega\rangle = 0, \quad (x,\omega),(x',\omega') \in B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d, \text{ with } x \neq x' .$$

Moreover, the family $\{M_\omega \chi_{\Omega+x} : \omega \in A^{-T}\mathbb{Z}^n\}$ is an orthogonal basis for $L^2(\Omega+x)$, $x \in \mathbb{R}^d$, by Theorem 2.4, so part 4 follows.

Now, if Ω is not a packing set for $B^{-T}\mathbb{Z}^d$, then there exists $x \in B^{-T}\mathbb{Z}^d$, $W = \Omega \cap (\Omega + x)$ with $m(W) > 0$. Note that $W - x = (\Omega - x) \cap \Omega \subseteq \Omega$. As Ω is a fundamental domain for $A\mathbb{Z}^d$, there exists $\{c_\omega\}_{\omega \in A^{-T}\mathbb{Z}^d}$ with

$$\chi_W + \chi_{W-x} = \sum_{\omega \in A^{-T}\mathbb{Z}^d} c_\omega M_\omega \chi_\Omega .$$

For $N \in \mathbb{N}$, consider

$$\begin{aligned} f_N &= \sum_{k=1}^{N} \sum_{\omega \in A^{-T}\mathbb{Z}^d} (-1)^k c_\omega T_{kx} M_\omega \chi_\Omega \\ &= \sum_{k=1}^{N} (-1)^k (\chi_{W+kx} + \chi_{W+(k-1)x}) = (-1)^N \chi_{W+Nx} - \chi_W . \end{aligned}$$

Clearly, $\|f_N\|_{L^2}^2 \leq 2m(W)$, but the coefficients $\{d_{x,\omega}\}_{(x,\omega)\in B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d}$ of the Gabor expansion of f_N in terms of $(\chi_\Omega, B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d)$ satisfy

$$\sum_{(x,\omega)\in B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d} |d_{x,\omega}|^2 = \sum_{k=1}^{N} \sum_{\omega \in A^{-T}\mathbb{Z}^d} |c_\omega|^2 = N \sum_{\omega \in A^{-T}\mathbb{Z}^d} |c_\omega|^2 \geq N m(W).$$

This implies that $(\chi_\Omega, B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d)$ is not a Riesz basis sequence. $\square$

Han and Wang's central result is the following.

Theorem 3.2 *Let $A\mathbb{Z}^d$ and $B\mathbb{Z}^d$ be two full-rank lattices in $\mathbb{R}^d$, such that $|\det A| = |\det B|$. Then there exists a measurable set Ω which is a fundamental domain for both $A\mathbb{Z}^d$ and $B\mathbb{Z}^d$. If $|\det A| \geq |\det B|$, then there exists a measurable set Ω, which is a packing set for $A\mathbb{Z}^d$ and a tiling set for $B\mathbb{Z}^d$.*

The combination of Proposition 3.1 and Theorem 3.2 provides us with the main result from [HW01, HW04]. The results were the first to prove the existence of Gabor frame windows for any separable lattice with density greater than or equal 1.

Theorem 3.3

1. *If $|\det A \det B| = 1$, then there exists $\Omega \subseteq \mathbb{R}^d$ such that $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*
2. *If $|\det A \det B| \geq 1$, then there exists $\Omega \subseteq \mathbb{R}^d$ such that $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is an orthogonal sequence for $L^2(\mathbb{R}^d)$.*
3. *If $|\det A \det B| \leq 1$, then there exists $\Omega \subseteq \mathbb{R}^d$ such that $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a tight Gabor frame for $L^2(\mathbb{R}^d)$ with frame bound $1/|\det B|$.*

Proof. Parts 2 and 3 follow directly from Theorem 3.2 and Lemma 3.1. Part 1 follows from part 2 and the observation that Ω tiles with respect to $A\mathbb{Z}^d$ and $B^{-T}\mathbb{Z}^d$, then $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is complete. $\square$

Proposition 3.1 states that if $\Omega \subseteq \mathbb{R}^d$ is a fundamental domain for $A\mathbb{Z}^d$, respectively $B^{-T}\mathbb{Z}^d$, then Ω must be a packing set for $B^{-T}\mathbb{Z}^d$, respectively $A\mathbb{Z}^d$, in order for $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ to be a frame, respectively a Riesz basis sequence. The interplay of the fundamental domain property of one lattice with the packing set property with respect to a second lattice is further illuminated by the following observation.

Proposition 3.4 *Let $\Omega \subseteq \mathbb{R}^d$.*

1. *If $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a frame for $L^2(\mathbb{R}^d)$, then Ω contains a fundamental domain of $A\mathbb{Z}^d$.*
2. *If Ω is a packing set for $B^{-T}\mathbb{Z}^d$, then $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if Ω contains a fundamental domain of $A\mathbb{Z}^d$.*
3. *If Ω is a packing set for $B^{-T}\mathbb{Z}^d$, then $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a tight frame for $L^2(\mathbb{R}^d)$ if and only if Ω is the union of $k \in \mathbb{N}$ disjoint fundamental domains of $A\mathbb{Z}^d$. The frame bound is then $k/|\det B|$.*

4. *If* $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ *is a Riesz basis sequence, then* Ω *contains a fundamental domain of* $B^{-T}\mathbb{Z}^d$.

5. *If* Ω *is a packing set for* $A\mathbb{Z}^d$*, then* $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ *is a Riesz basis sequence if and only if* Ω *contains a fundamental domain of* $B^{-T}\mathbb{Z}^d$.

6. *If* Ω *is a packing set for* $A\mathbb{Z}^d$*, then* $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ *is an orthogonal sequence if and only if* Ω *is the union of a finite number of fundamental domains of* $B^{-T}\mathbb{Z}^d$.

Proof. Theorem 2.7 implies that it suffices to show parts 1, 5, and 6. The first statement is trivial, as else $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ would not span $L^2(\mathbb{R}^d)$.

In the remaining parts, we assume that $\Omega \subseteq \mathbb{R}^d$ is a packing set for $A\mathbb{Z}^d$, hence, it suffices to show that $\{M_\omega \chi_\Omega\}_{\omega \in B\mathbb{Z}^d}$ is a Riesz basis sequence if and only if Ω contains a fundamental domain of $B^{-T}\mathbb{Z}^d$, respectively an orthogonal sequence if and only if Ω is the union of a finite number of fundamental domains of $B^{-T}\mathbb{Z}^d$.

If Ω contains a fundamental domain of $B^{-T}\mathbb{Z}^d$, then clearly $\{M_\omega \chi_\Omega\}_{\omega \in B\mathbb{Z}^d}$ is a Riesz basis sequence [Chr03]. If Ω does not contain a fundamental domain of $B^{-T}\mathbb{Z}^d$, then $\sum_{\omega \in B^{-T}\mathbb{Z}^d} \chi_{\Omega+\omega} = 0$ on a set $W \subseteq B^{-T}[0,1)^d$ with $m(W) > 0$. Note that we have also $\sum_{\omega \in B^{-T}\mathbb{Z}^d} \chi_{\Omega+\omega} = 0$ on a set $W + x$ for any $x \in B^{-T}\mathbb{Z}^d$.

As $B^{-T}[0,1)^d$ is a fundamental domain for $B^{-T}\mathbb{Z}^d$, there exist $\{c_\gamma\}_{\gamma \in B\mathbb{Z}^d} \neq 0$ such that $\chi_W = \sum c_\gamma M_\gamma \chi_{B^{-T}[0,1)^d}$. Then $\sum c_\gamma M_\gamma \chi_{\mathbb{R}^d} = \sum_{\omega \in B^{-T}\mathbb{Z}^d} \chi_{W+\omega}$ and

$$0 = \chi_\Omega \Big(\sum_{\omega \in B^{-T}\mathbb{Z}^d} \chi_{W+\omega} \Big) = \sum_{\gamma \in B\mathbb{Z}^d} c_\gamma M_\gamma \chi_\Omega.$$

We conclude that $\{M_\omega \chi_\Omega\}_{\omega \in B\mathbb{Z}^d}$ is not a Riesz basis.

Now, if Ω is the finite union of fundamental domains of $B^{-T}\mathbb{Z}^d$, then clearly $\{M_\omega \chi_\Omega\}_{\omega \in B\mathbb{Z}^d}$ is an orthogonal sequence. On the other side, if $\{M_\omega \chi_\Omega\}_{\omega \in B\mathbb{Z}^d}$ is an orthogonal sequence, then for $\omega \neq 0$,

$$\begin{aligned} 0 &= \int M_\omega \chi_\Omega(t)\, dt = \int e^{2\pi i \omega t} \chi_\Omega(t)\, dt \\ &= \sum_{p \in B^{-T}\mathbb{Z}^d} \int_{B^{-T}[0,1)^d + p} e^{2\pi i \omega t} \chi_\Omega(t)\, dt \\ &= \int_{B^{-T}[0,1)^d} \sum_{p \in B^{-T}\mathbb{Z}^d} e^{2\pi i \omega (t-p)} \chi_\Omega(t-p)\, dt \\ &= \int_{B^{-T}[0,1)^d} e^{2\pi i \omega t} \sum_{p \in B^{-T}\mathbb{Z}^d} \chi_\Omega(t-p)\, dt\,. \end{aligned} \tag{5}$$

Hence, $\sum_{p \in B^{-T}\mathbb{Z}^d} \chi_\Omega(t-p)$ is an integer-valued constant function, that is, $\sum_{p \in B^{-T}\mathbb{Z}^d} \chi_\Omega(t-p) = k \in \mathbb{N}$. This implies that almost every point in $\mathbb{R}^d$ is covered by k translates of Ω. Hence, Ω is the union of k fundamental domains. In fact,

we can find a fundamental domain $\Omega_1 \subseteq \Omega$, remove it as shown below, and then continue inductively.

As long as $\sum_{p\in B^{-T}\mathbb{Z}^d} \chi_\Omega(t-p) > 1$ on a set of nonzero measure, there exists $q \in B^{-T}\mathbb{Z}^d$ such that $\Omega + q \cap \Omega$ has nonzero measure. Set $\Omega' = \Omega \setminus (\Omega + q)$. Now,

$$\bigcup_{p\in B^{-T}\mathbb{Z}^d} (\Omega + p) = \bigcup_{p\in B^{-T}\mathbb{Z}^d} (\Omega' + p)$$

follows from $\bigcup_{\ell\in\mathbb{Z}} \big(\Omega + \ell q\big) = \bigcup_{\ell\in\mathbb{Z}} \big(\Omega' + \ell q\big)$, which in turn follows from $\Omega \subseteq \bigcup_{\ell\in\mathbb{Z}} \big(\Omega' + \ell q\big)$. But if $x \in \Omega \setminus \Big(\bigcup_{\ell\in\mathbb{Z}} \big(\Omega' + \ell q\big)\Big)$, then $x \in \Omega + \ell q$ for all $\ell \in \mathbb{N}$, a contradiction. $\square$

Remark 3.5 For $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ to be a frame for $L^2(\mathbb{R}^d)$ it is not necessary that $\Omega \subseteq \mathbb{R}^d$ is a packing set for $B^{-T}\mathbb{Z}^d$. In fact, $(\chi_{[0,3/2)},\ 1/2\mathbb{Z} \times \mathbb{Z})$ is a frame [Jan03].

We close with a simple, but interesting example.

Example 3.6 Let W be a fundamental domain for $1/2\mathbb{Z} \times \mathbb{Z}$, $(x, y) \in \mathbb{R}^2$, and $\Omega = W + (x, y) \cup W$. Then $(\chi_\Omega,\ 1/2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$ is a frame for $L^2(\mathbb{R}^2)$ if Ω is a packing set for $(\mathbb{Z} \times \mathbb{Z})^{-T} = \mathbb{Z} \times \mathbb{Z}$. A simple computation shows that this is the case if and only if $W \cap W \pm (k + x, \ell + y) = \varnothing$ for all $(k, \ell) \in \mathbb{Z}^2$. If Ω is a packing set for $\mathbb{Z} \times \mathbb{Z}$, then $(\chi_\Omega,\ \frac{1}{2}\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z})$ is a tight frame for $L^2(\mathbb{R}^2)$ if and only if $x \in (2\mathbb{Z} + 1)$ and $y \in \mathbb{Z}$. For a detailed discussion see [PR10]

4 Existence of smooth and compactly supported Gabor frame windows

Han and Wang's construction of Gabor orthonormal bases for separable lattices is a landmark result in Gabor analysis. The drawback of their approach lies in the fact that the constructed functions are discontinuous and may not even decay at infinity. In this section, we will construct nonnegative, compactly supported, and smooth window functions for a class of separable lattices.

We begin this section with a simple result to indicates a limitation of Theorem 3.3 as well as the direction which we will take. Its derivation is included in the appendix.

Proposition 4.1 *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ with $d(\Lambda) > 1$. Let Ω be a fundamental domain for $A\mathbb{Z}^d$ and a packing set for $B^{-T}\mathbb{Z}^d$. If $g \in C(\mathbb{R}^d)$ and $\operatorname{supp} g = \Omega$, then the Gabor system (g, Λ) is complete, but not a frame for $L^2(\mathbb{R}^d)$.*

In view of Proposition 4.1 we have to consider window functions in $C_c^\infty(\mathbb{R}^d)$ whose support extends beyond the fundamental domain Ω of $A\mathbb{Z}^d$. Recall that $\Omega_\epsilon = \{x + y, x \in \Omega, \|y\|_2 \le \epsilon\}$. Further, $\phi \in C_c^\infty(\mathbb{R}^d)$ is nonnegative and satisfies $\operatorname{supp} \phi \subseteq B_1(0) = \{x \in \mathbb{R}^d : \|x\|_2 \le 1\}$ and $\int \phi = 1$. Further $\phi_\epsilon(x) = 1/\epsilon\, \phi(x/\epsilon)$.

Theorem 4.2

1. *If there exists $\Omega \subseteq \mathbb{R}^d$, $\epsilon > 0$, such that Ω is a fundamental domain for $A\mathbb{Z}^d$ and Ω_ϵ is a packing set for $B^{-T}\mathbb{Z}^d$, then $(\sqrt{\chi_\Omega * \phi_\epsilon}, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound $1/|\det B|$.*

2. *Suppose there exists $\Omega \subseteq \mathbb{R}^d$, $\epsilon > 0$, such that Ω is a fundamental domain for $B^{-T}\mathbb{Z}^d$ and Ω_ϵ is a packing set for $A\mathbb{Z}^d$, then $(\sqrt{\chi_\Omega * \phi_\epsilon}, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is an orthonormal system.*

Note that the conditions on Ω imply $d(\Lambda) < 1$ in the first statement and $d(\Lambda) > 1$ in the second statement of Theorem 4.2.

Proof. We shall prove the second statement, the first statement follows then from Theorem 2.7 and the observation that

$$\|\sqrt{\chi_\Omega * \phi_\epsilon}\|_{L^2}^2 = \int |\chi_\Omega * \phi_\epsilon| = \int \chi_\Omega = m(\Omega) = |\det A|.$$

Following the proof of Theorem 3.3, we obtain that (χ_Ω, Λ) is an orthonormal system. Moreover, $\operatorname{supp} \chi_\Omega * \phi_\epsilon \subseteq \Omega_\epsilon$, and as Ω_ϵ is a packing set for $A\mathbb{Z}^d$, we maintain $\pi(k,\ell)\sqrt{\chi_\Omega * \phi_\epsilon}$ is orthogonal to $\pi(k',\ell')\chi_\Omega * \phi_\epsilon$ if $k \neq k'$. It remains to show that $\{\pi(0,\ell')\chi_\Omega * \phi_\epsilon\}$ is orthogonal. But this follows as for $\ell \neq \ell'$

$$\begin{aligned}\langle \pi(0,\ell)\sqrt{\chi_\Omega * \phi_\epsilon}\ ,\ \pi(0,\ell')\sqrt{\chi_\Omega * \phi_\epsilon}\rangle &= \int e^{2\pi i(\ell-\ell')x}\chi_\Omega * \phi_\epsilon(x)\,dx \\ &= (\chi_\Omega * \phi_\epsilon)\widehat{\ }(\ell'-\ell) = \widehat{\chi_\Omega}(\ell'-\ell)\widehat{\phi_\epsilon}(\ell'-\ell) = 0\,.\end{aligned}$$

□

To construct sets that allow for the application of Theorem 4.2, we turn to lattices which have starshaped common fundamental domains.

Proposition 4.3 *If $0 < |\det A| < |\det B|$ and $|\det B/\det A|^{1/d}A\mathbb{Z}^d$ and $B\mathbb{Z}^d$ have a bounded and star-shaped common fundamental domain, then exists $\Omega \subseteq \mathbb{R}^d$ such that Ω_ϵ is a packing set for $B\mathbb{Z}^d$ and Ω is a tiling set for $A\mathbb{Z}^d$.*

Proof. We let $\tilde{A} = |\det B/\det A|^{1/d}A\mathbb{Z}^d$, and obtain $|\det \tilde{A}| = |\det B|$. By Theorem 3.2 there exists a measurable set Ω' which is a common fundamental domain for $\tilde{A}\mathbb{Z}^d$ and $B\mathbb{Z}^d$, and by hypothesis, we can assume Ω' is star-shaped. We claim that there exists a fundamental domain Ω for $A\mathbb{Z}^d$ such that $\Omega_\epsilon \subset \Omega'$. By our hypothesis, we choose a point $N \in \Omega'$ such that for all $P \in \Omega'$, the segment $\overrightarrow{NP}$ is contained in the interior of Ω'. Without loss of generality we may take N to be the origin (Figure 1). We apply a dilation with center N and coefficient $|\det B/\det A|^{-1/d}$ to Ω' and obtain a set Ω with $\Omega \cap \Omega' = \Omega$. Clearly, Ω is a fundamental domain for the lattice $A\mathbb{Z}^d$. □

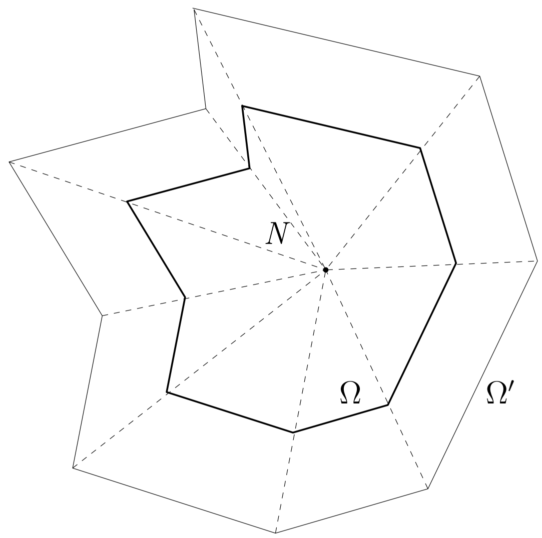

Figure 1: Ω is the scaled copy of Ω' under dilation with center N.

Corollary 4.4 *For $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ with $d(\Lambda) > 1$ suppose that the lattices $d(\Lambda)^{1/d} A\mathbb{Z}^d$ and $B^{-T}\mathbb{Z}^d$ have a bounded and star-shaped common fundamental domain Ω, then exists a nonnegative $g \in \mathbb{C}_c^\infty(\mathbb{R}^d)$ with (g, Λ) being a tight frame for $L^2(\mathbb{R}^d)$.*

Proof. The result follows from Theorem 4.2, part 2, and Proposition 4.3 where we replace B by B^{-T} and note that

$$1 < d(\Lambda) = \left|\frac{1}{\det A \det B}\right| = \left|\frac{\det B^{-T}}{\det A}\right|$$

□

Corollary 4.4 can be extended to a class of upper and lower-block triangular lattices.

Corollary 4.5 *If the lattices $A\mathbb{Z}^d$ and $|\det A \det B|^{1/d} B^{-T}\mathbb{Z}^d$ have a bounded and star-shaped common fundamental domain Ω' and if D is such that DA^{-1} symmetric, then exists $g \in C_c^\infty(\mathbb{R}^d)$ such that $(g, \left(\begin{smallmatrix} A & 0 \\ D & B \end{smallmatrix}\right)\mathbb{Z}^{2d})$ is a frame for $L^2(\mathbb{R}^d)$.*

Proof. Let $T = \left(\begin{smallmatrix} I & 0 \\ -DA^{-1} & I \end{smallmatrix}\right)$. Then $T\left(\begin{smallmatrix} A & 0 \\ D & B \end{smallmatrix}\right)\mathbb{Z}^d = A\mathbb{Z}^d \times B\mathbb{Z}^d$ is separable and fulfills the conditions of Corollary 4.4, so there exists $\tilde{g} \in C_c^\infty(\mathbb{R}^d)$ such that $(\tilde{g}, T\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. Since T is symplectic, by Theorem 2.10 there exists $g \in \mathcal{S}(\mathbb{R}^d)$ such that (g, Λ) is a frame for $L^2(\mathbb{R}^d)$. Furthermore, $g \in C_c^\infty(\mathbb{R}^d)$ because the metaplectic operator associated to T is a multiplication by a chirp, which preserves the compact support of $\tilde{g}$ [Fol89]. □

5 Bivariate examples

In this section we provide several examples which illustrate the geometric criteria established above for a family of matrices in the case $d = 2$.

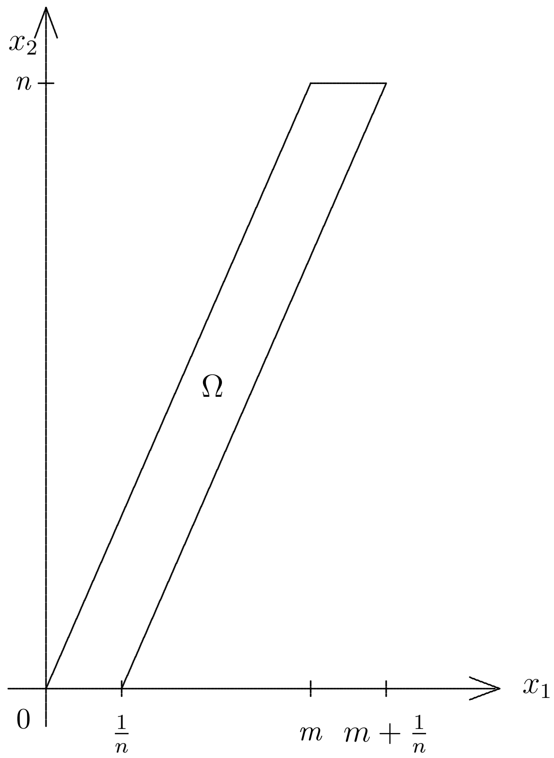

Figure 2: The set Ω constructed in Proposition 5.1.

Proposition 5.1 *Let $q \in \mathbb{Q}^+$ and m, n co-prime integers such that $q = m/n$. There exists a common convex fundamental domain for $\mathbb{Z}^2$, $\begin{pmatrix} q & 0 \\ 0 & 1/q \end{pmatrix} \mathbb{Z}^2$ and $\begin{pmatrix} 1/n & m \\ 0 & n \end{pmatrix} \mathbb{Z}^2$. Similarly, the lattices $\mathbb{Z}^2$, $\begin{pmatrix} n/m & 0 \\ 0 & m/n \end{pmatrix} \mathbb{Z}^2$ and $\begin{pmatrix} n & 0 \\ m & 1/n \end{pmatrix} \mathbb{Z}^2$, have a common convex fundamental domain.*

Proof. We shall only prove the first assertion, the second one follows analogously.

Let $\Omega = \begin{pmatrix} 1/n & m \\ 0 & n \end{pmatrix} [0,1)^2$. Ω is a convex set and $m(\Omega) = 1$. Suppose that there exists $(k,l)^T \neq \vec{0} \in \mathbb{Z}^2$ such that $\{\Omega + (k,l)^T\} \cap \Omega \neq \varnothing$. Then there exist points $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in [0,1)^2$ such that $\frac{\alpha_1}{n} + m\beta_1 + k = \frac{\alpha_2}{n} + m\beta_2$ and $n\beta_1 + l = n\beta_2$. Therefore, $\beta_2 - \beta_1 = \frac{l}{n}$, which implies that $\alpha_1 - \alpha_2 = ml - kn \in \mathbb{Z}$. Since $0 \leq \alpha_1, \alpha_2 < 1$, necessarily $\alpha_1 = \alpha_2$, and also that $\beta_2 - \beta_1 = \frac{k}{m}$. Since $\gcd(m,n) = 1$ and $0 \leq \beta_2 - \beta_1 < 1$, this is possible only if $k = l = 0$. Thus $(\Omega + \mathbb{Z}^2 \backslash \{0\}) \cap \Omega = \varnothing$. Since $m(\Omega) = 1$, Ω is a fundamental domain for $\mathbb{Z}^2$.

A similar proof shows that Ω is also a fundamental domain for the lattice $\begin{pmatrix} m/n & 0 \\ 0 & n/m \end{pmatrix} \mathbb{Z}^2$. $\square$

To illustrate strength and weakness of our method, we shall consider the following, apparently simple example.

Corollary 5.2 *If $\Lambda = a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z} \times d\mathbb{Z}$ with $d(\Lambda) = abcd < 1$ satisfies*

$$(1)\ ac < 1, bd < 1, \quad or \quad (2)\ abcd < 1/2, \quad or \quad (3)\ \sqrt{\frac{ac}{bd}} \in \mathbb{Q}$$

then there exists $g \in C_c^\infty(\mathbb{R}^2)$ such that (g, Λ) is a tight frame for $L^2(\mathbb{R}^2)$.

Proof. (1) If $ac < 1$ and $bd < 1$, then any $g_1 \in C_c^\infty(\mathbb{R})$ with $\chi_{[0,a]} \leq g_1 \leq \chi_{[(ac-1)/(2c),(ac+1)/(2c)]}$ and $g_2 \in C_c^\infty(\mathbb{R})$ with $\chi_{[0,b]} \leq g_2 \leq \chi_{[(bd-1)/(2d),(bd+1)/(2d)]}$ guarantees that $(g_1, a\mathbb{Z} \times c\mathbb{Z})$ and $(g_2, b\mathbb{Z} \times d\mathbb{Z})$ are frames. A simple tensor argument then implies that $(g_1 \otimes g_2, a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z} \times d\mathbb{Z})$ is a frame for $L^2(\mathbb{R}^d)$. Note that the same line of argument shows that if either $ac > 1$ or $bd > 1$, then exists no $g_1, g_2 \in C_c^\infty(\mathbb{R})$ with $(g_1 \otimes g_2, a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z} \times d\mathbb{Z})$ is a frame for $L^2(\mathbb{R}^d)$ [PR10].
(2) It suffices to consider $abcd < 1/2$, and $ac > 1$ or $bd > 1$, as else, (1) would apply. Without loss of generality, we consider $ac > 1$, and, hence $bd < 1/2$. Moreover, applying Theorem 2.10 with the symplectic matrix $M = \mathrm{diag}(c, d, 1/c, 1/d)$ implies that the existence of $g \in C_c^\infty(\mathbb{R}^2)$ with (g, Λ) being a frame for $L^2(\mathbb{R}^2)$ follows from the respective statement for $\Lambda' = \mathrm{diag}(c, d, 1/c, 1/d)\Lambda = \mathrm{diag}(ac, bd, 1, 1)\mathbb{Z}^4$. With $\Omega = \left(\begin{smallmatrix} ac & 0 \\ 1/2 & bd \end{smallmatrix}\right)[0,1)^2$, $\epsilon = \frac{1-2abcd}{4ac}$, any $g \in C_c^\infty(\mathbb{R}^2)$ with

$$\chi_\Omega \leq g \leq \chi_{\Omega_\epsilon}$$

has the property that (g, Λ') is a frame for $L^2(\mathbb{R}^2)$.
(3) Theorem 5.4 applies whenever there exist $m, n \in \mathbb{Z}, \alpha \in \mathbb{R}$ such that $\alpha m^2 = ac$ and $\alpha n^2 = bd$, which is equivalent to $\sqrt{\frac{ac}{bd}} \in \mathbb{Q}$. □

The conditions on Λ presented in Proposition 5.2 are not necessary for the existence of $g \in C_c^\infty(\mathbb{R}^2)$ with (g, Λ) being a Gabor frame for $L^2(\mathbb{R}^d)$. In fact, if for $g \in C_c^\infty(\mathbb{R}^2)$, $(g, M\mathbb{Z}^4)$ is a Gabor frame for $L^2(\mathbb{R}^2)$, then exists an open neighborhood U of M in $GL(\mathbb{R}^4)$, such that $(g, M'\mathbb{Z}^4)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ for all $M' \in U$ [FK04]. For results where rationality of lattices plays a central role, see results known as Janssen's tie [Jan03].

Below, we show that the condition $abcd < 1$ and the use of diagonal matrices with rational entries in Corollary 5.2 is critical for our method to be applicable. This clearly illustrates the limitations of the method described in Theorem 3.3.

Proposition 5.3 *There exists no fundamental domain Ω for $\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ with Ω_ϵ, $\epsilon > 0$, is a packing set for $\mathbb{Z} \times \mathbb{Z}$. Consequently, there exists no common star-shaped fundamental domain for $\mathbb{Z}^2$ and $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\mathbb{Z}^2$.*

Proof. Suppose that Ω is a tiling set for $\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. If Ω_ϵ is a packing set for $\mathbb{Z} \times \mathbb{Z}$, then $\{\Omega + (m, n) : m, n \in \mathbb{Z}\}$ have no boundary points in common. Hence all sets $\{\Omega + (m, \frac{n}{2}) : m, n \in \mathbb{Z}\}$ with common boundary point with Ω must be of the form $\Omega + (m, n + \frac{1}{2})$. Clearly, there must be two such sets in the corona of Ω which have a common boundary points. But this is a contradiction as the system

$\{\Omega + (m, n + \frac{1}{2}) : m, n \in \mathbb{Z}\}$ is a translate of the system $\{\Omega + (m, n) : m, n \in \mathbb{Z}\}$ and hence all of its members should have disjoint boundaries.

To obtain the second assertion, assume that there exists a compact star-shaped set Ω' serving as a common fundamental domain for both lattices. Then there exists $x \in \mathbb{R}^2, \epsilon > 0$ such that for $\Omega = D_{\frac{1}{\sqrt{2}}}\Omega' + x$, we have $\Omega \subseteq \Omega_\epsilon \subseteq \Omega'$. Note that Ω is a tiling set for $\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}\mathbb{Z}^2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\mathbb{Z}^2$, contradicting the first assertion.

$\square$

Theorem 5.4 *Let $m, n \in \mathbb{Z}$ be relatively prime. Let $\Lambda = A\mathbb{Z}^2 \times B\mathbb{Z}^2$ be a lattice in $\mathbb{R}^4$. Whenever B^TA is of the form*

1. αI, $|\alpha| < 1$;
2. $\begin{pmatrix} m^2\alpha & 0 \\ 0 & n^2\alpha \end{pmatrix}$, *where* $|\alpha| < (mn)^{-1}$;
3. $\begin{pmatrix} \alpha & mn\alpha \\ 0 & n^2\alpha \end{pmatrix}$, *where* $|\alpha| < n^{-1}$; *or*
4. $\begin{pmatrix} n^2\alpha & 0 \\ mn\alpha & \alpha \end{pmatrix}$, *where* $|\alpha| < n^{-1}$,

then there exists a function $g \in C_c^\infty(\mathbb{R}^2)$ such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^2)$.

Proof. We have

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \underbrace{\begin{pmatrix} B^{-T} & 0 \\ 0 & B \end{pmatrix}}_{M} \begin{pmatrix} B^TA & 0 \\ 0 & I \end{pmatrix}.$$

This shows that $\Lambda = M((B^TA)\mathbb{Z}^2 \times \mathbb{Z}^2)$ with M symplectic. Since $|\det B^TA| \leq 1$, we can rescale $B^TA\mathbb{Z}^2$ to make its density 1. Then $|\det B^TA|^{-1/2}B^TA\mathbb{Z}^2$ is respectively of the form

$$\mathbb{Z}^2, \begin{pmatrix} m/n & 0 \\ 0 & n/m \end{pmatrix}\mathbb{Z}^2, \begin{pmatrix} 1/n & m \\ 0 & n \end{pmatrix}\mathbb{Z}^2, \begin{pmatrix} n & 0 \\ m & 1/n \end{pmatrix}\mathbb{Z}^2$$

Proposition 5.1 assures the existence of a common convex fundamental domain for $(B^TA)\mathbb{Z}^2$ and $\mathbb{Z}^2$ accordingly. By Theorem 3.3 there exists $g' \in C_c^\infty(\mathbb{R}^2)$ such that $(g', (B^TA)\mathbb{Z}^2 \times \mathbb{Z}^2)$ is a Gabor frame for $L^2(\mathbb{R}^2)$. The matrix M is symplectic, and its associated metaplectic operator $\mu(M)$ from Theorem 2.10 is the dilation $(\mu(M)h)(x) = |\det B|^{-\frac{1}{2}}h(B^{-1}x)$, [Fol89]. Hence, $g = \mu(M)^*g' \in C_c^\infty(\mathbb{R}^2)$ and (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^2)$. $\square$

Note that it is not known for which α, β there exists $g \in \mathcal{S}(\mathbb{R}^2)$ such that $(g, \mathbb{Z}^2 \times \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is Gabor frame for $L^2(\mathbb{R}^d)$. If $g_0(x) = e^{-\pi\|x\|_2^2}$ is a Gaussian, then $(g_0, \mathbb{Z}^2 \times \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is a frame for $L^2(\mathbb{R}^d)$ if $\alpha, \beta < 1$ [PR10].

6 Appendix: Proof of Proposition 4.1

The operator

$$S_{g,\Lambda}f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)g, \quad f \in L^2(\mathbb{R}^d)$$

is called a Gabor frame operator. It is positive and self-adjoint if (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ [Grö01, Chr03]. Gabor frames possess a very useful reconstruction formula:

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)\gamma = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle\, \pi(\lambda)g, \quad f \in L^2(\mathbb{R}^d)$$

with $\gamma = S_{g,\Lambda}^{-1}g$ being the so-called canonical dual window [Grö01, Chr03].

$S_{g,A\mathbb{Z}^d\times B\mathbb{Z}^d}$ can be represented in matrix form [Wal92, RS97]. For that purpose, we define the bi-infinite cross-ambiguity Gramian matrix

$$\begin{aligned} &\mathbf{G}(x) = (G_{jk}(x))_{j,k\in\mathbb{Z}^d} : \\ &G_{jk}(x) = |\det B|^{-1} \sum_{\ell\in\mathbb{Z}^d} \overline{g(x - B^{-T}k - A\ell)}g(x - B^{-T}j - A\ell). \end{aligned} \tag{6}$$

Below, $W(\mathbb{R}^d)$ denotes the Wiener space, consisting of all functions such that the norm

$$\|f\|_W = \sum_{k\in\mathbb{Z}^d} \|f \cdot T_k\chi_{[0,1)^d}\|_\infty$$

is finite [Chr03].

Proposition 6.1 *Let $g \in W(\mathbb{R}^d)$. Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full-rank lattice in $\mathbb{R}^{2d}$. For $f, h \in L^2(\mathbb{R}^d)$, define the sequences*

$$\mathbf{f}(x) := \{f(x - B^{-T}j) : j \in \mathbb{Z}^d\}, \quad \mathbf{h}(x) := \{h(x - B^{-T}k) : k \in \mathbb{Z}^d\}.$$

Then the following holds:

$$\langle S_{g,\Lambda}f, h\rangle = \int_{B^{-T}[0,1)^d} \langle \mathbf{G}(x)\mathbf{f}(x), \mathbf{h}(x)\rangle dx$$

for all $f, h \in L^2(\mathbb{R}^d)$. $S_{g,\Lambda}$ is a bounded operator on $L^2(\mathbb{R}^d)$ if and only if there exists $b > 0$ such that $\mathbf{G}(x) \le bI_{\ell^2}$ for almost all $x \in \mathbb{R}^d$. Also $S_{g,\Lambda}$ is a boundedly invertible operator on $L^2(\mathbb{R}^d)$ if and only if there exists $a > 0$ such that $\mathbf{G}(x) \ge aI_{\ell^2}$ for almost all $x \in \mathbb{R}^d$.

Note that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$ if and only if $S_{g,\Lambda}$ is bounded and boundedly invertible on $L^2(\mathbb{R}^d)$.

Proof of Proposition 4.1. Let Ω be a fundamental domain for $A\mathbb{Z}^d$ and packing set for $B^{-T}\mathbb{Z}^d$, $g \in C(\mathbb{R}^d)$ with $\operatorname{supp} g = \Omega$. Let $f \in L^2(\mathbb{R}^d)$. Denote by f_k

the restriction of f to $\Omega + Ak$, $k \in \mathbb{Z}^d$. Thus $\|f\|_2^2 = \sum_{k\in\mathbb{Z}^d} \|f_k\|_2^2$, where $f_k \in L^2(\Omega + Ak)$.

To prove completeness, suppose there exists $f \in L^2(\mathbb{R}^d)$ such that

$$\langle f, M_{Bl}T_{Ak}g\rangle = 0, \quad k, l \in \mathbb{Z}^d. \tag{7}$$

However, because $\operatorname{supp} g = \Omega$, for a fixed $k \in \mathbb{Z}^d$, (7) is the Fourier coefficient $(f_k \cdot T_{Ak}g)\widehat{\ }(Bl)$. From the Fourier series expansion

$$(f_k \cdot T_{Ak}g)(t) = \sum_{l\in\mathbb{Z}^d} (f_k \cdot T_{Ak}g)\widehat{\ }(Bl)\, e^{2\pi i Bl\cdot t}$$

we see that $f_k \cdot T_{Ak}g$ is identically 0 almost everywhere on $\Omega + Ak$. Because g does not vanish on a subset of Ω of positive measure, $f_k = 0$ almost everywhere on $\Omega + Ak$ for all k. Therefore $f = 0$ almost everywhere on $\mathbb{R}^d$ and completeness of (g, Λ) is shown.

Assume now that $(g, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. We analyze the structure of the associated cross-ambiguity matrix $\mathbf{G}(x)$. If $j \neq k$, then

$$\operatorname{supp} g(x - B^{-T}k - Al)g(x - B^{-T}j - Al) \subseteq Al + [(\Omega + B^{-T}k) \cap (\Omega + B^{-T}j)].$$

Since $m((\Omega + B^{-T}k) \cap (\Omega + B^{-T}j)) = 0$, for almost all x we have $G_{kj}(x) = 0$ for $j \neq k$. Thus the matrix $\mathbf{G}(x)$ given by (6) is diagonal for almost all x. Moreover,

$$G_{00}(x) = \sum_{l\in\mathbb{Z}^d} |g(x - Al)|^2 = |g(x)|^2,$$

since $\operatorname{supp} g = \Omega$. When $\mathbf{c} = \{\delta_0(n)\}_{n\in\mathbb{Z}^d}$, Proposition 6.1 implies that $a \leq \langle \mathbf{G}(x)\mathbf{c}, \mathbf{c}\rangle \leq b$, because S_g is a bounded and invertible operator on $L^2(\mathbb{R}^d)$. Therefore, the associated matrix $\mathbf{G}(x)$ shares these properties for almost every x. But then $\langle \mathbf{G}(x)\mathbf{c}, \mathbf{c}\rangle = G_{00}(x) = |g(x)|^2$, which in turn implies $a \leq |g(x)|^2 \leq b$ on Ω, contradicting the continuity of g on $\mathbb{R}^d$. $\square$

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